# Harmonic functions. Poisson's formula. Schwarz's theorem

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<u>Def</u>. Let $u \in C^2(\Omega)$ (twice continuously <u>real</u> differentiable)
$u$ is called <u>harmonic</u> if $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \operatorname{div}(\nabla u) = 0$.
<u>Notation</u>. $Harm(\Omega)$. We'll consider real-valued harmonic functions.

As for holomorphic: $u \in Harm(S)$ means $\exists \mathcal{O} \supset S$-open, $u \in Harm(\mathcal{O})$

<u>Reminder</u>. $f \in A(\Omega) \Rightarrow \operatorname{Re} f, \operatorname{Im} f \in Harm(\Omega)$.
Follows from Cauchy-Riemann.

Is the opposite true?
Not always: $\log|z| \in Harm(\mathbb{C} \setminus 0)$ ($\forall z$ $\log|z| = \operatorname{Re}\log z$ locally, i.e. in $B(z,|z|)$ there is a branch of logarithm).

But if $\exists f \in A(\mathbb{C} \setminus 0)$; $\operatorname{Re} f = u$
then $f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{1}{z}$, and $\oint\limits_{\{|z|=1\}} \frac{dz}{z} \neq 0 \Rightarrow \frac{1}{z}$ has no antiderivative! Contradiction!
Cauchy-Riemann

But true in simply-connected regions:
<u>Theorem</u>. Let $\Omega$ be a simply-connected region.
Let $u \in Harm(\Omega)$. Then $\exists f \in A(\Omega)$; $u = \operatorname{Re} f$
If $u = \operatorname{Re} f_1 = \operatorname{Re} f_2$ then $f_1 - f_2 \equiv const \in i\mathbb{R}$.

<u>Corollary</u> $u \in Harm(\Omega)$ $(\forall \Omega) \Rightarrow u \in C^\infty(\Omega)$.
<u>Proof</u> (Theorem $\Rightarrow$ Corollary).
Let $z \in \Omega$, $\exists B(z,r) \subset \Omega$. $B(z,r)$ - simply connected.
So $\exists f \in A(B(z,r))$: $u = \operatorname{Re} f$. $f \in C^\infty \Rightarrow u \in C^\infty$

<u>Proof</u> (of Theorem).
Let $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = u_1 + iv_1$.

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial v_1}{\partial y} \\ \frac{\partial u_1}{\partial y} &= \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial v_1}{\partial x} \end{aligned}\right\} \Rightarrow g \in A(\Omega).$$

$\Omega$ - simply connected. So $\exists f \in A(\Omega)$: $g(z) = f'(z)$.
Let $f(z) = U(z) + iV(z)$.
$$g(z) = \frac{\partial U}{\partial x} - i\frac{\partial V}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \Rightarrow \frac{\partial (U-u)}{\partial x} = \frac{\partial (U-u)}{\partial y} = 0 \Rightarrow U = u + const.$$
So $\operatorname{Re}(f - const) = u$.
If $\operatorname{Re} f_1 = \operatorname{Re} f_2 \Leftrightarrow \operatorname{Re}(f_1 - f_2) \equiv 0 \Rightarrow f_1 = f_2 + ic$.